Comment on "Steady-state properties of a totally asymmetric exclusion process with periodic structure"

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Lakatos *et al.* [Phys. Rev. E **71**, 011103 (2005)] have studied a totally asymmetric exclusion process that contains periodically varying movement rates. They have presented a cluster mean-field theory for the problem. We show that their cluster mean-field theory leads to redundant equations. We present a mean-field analysis in which there is no redundant equation.

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Recently, Lakatos *et al.* used the cluster mean field to analyze a totally asymmetric exclusion process (TASEP) with periodic structure [1]. In the model of TASEP with periodic structure, the TASEP is generalized to include two internal hopping rates, p_1 and p_2. We consider periodic boundary conditions, and suppose there are N particles in the lattice with $2L$ sites. The density is therefore $\sigma = N/(2L)$. In each time step, a particle is chosen randomly. If it is on an even (odd) site, then it hops forward by one site with probability p_1 (p_2) provided the target site is empty.

The model corresponds to $T=2$ in Ref. [1], where T is the period of the periodic structure. In Ref. [1], Lakatos *et al.* consider the pair probability $P(x_i, x_{i+1})$ [i.e., the probability of finding a p_1 site with occupancy x_i ($x_i=0$ if site i is empty, $x_i=1$ if site i is occupied), followed by a p_2 site with occupancy x_{i+1}], and the pair probability $Q(x_i, x_{i+1})$ (i.e., the probability of finding a p_2 site with occupancy x_i followed by a p_1 site with occupancy x_{i+1}). They believe "the time evolution of the occupancy state of any two adjacent sites will depend on the two sites themselves along with the pair of sites immediately to the left or the right of the two site group." Thus, they write out the master equation for the two-site probability $P(0,0)$ as follows:

$$\frac{dP(0,0)}{dt} = -p_2[P(0,1,0,0) + P(1,1,0,0)] + p_2[P(0,1,0,0) + P(0,1,0,1)]. \tag{1}$$

They assume that each pair of (p_1, p_2) sites behaves as a statistically independent unit and they decompose the probabilities into products of pair probabilities, $P(x_i, x_{i+1}, x_{i+2}, x_{i+3}) = P(x_i, x_{i+1})P(x_{i+2}, x_{i+3})$. As a result, Eq. (1) is reformulated into

$$\frac{dP(0,0)}{dt} = p_2[P(0,1)^2 - P(1,1)P(0,0)]. \tag{2}$$

In the steady state, $\frac{dP(0,0)}{dt}=0$, thus

$$p_2[P(0,1)^2 - P(1,1)P(0,0)] = 0. \tag{3}$$

Let σ_1 and σ_2 denote the densities at sites p_1 and p_2. It is clear that

$$\sigma_1 = P(1,0) + P(1,1), \tag{4}$$

$$\sigma_2 = P(0,1) + P(1,1). \tag{5}$$

For $Q(x_i, x_j)$, one has the counterparts of Eqs. (3)–(5),

$$p_1[Q(0,1)^2 - Q(1,1)Q(0,0)] = 0, \tag{6}$$

$$\sigma_1 = Q(0,1) + Q(1,1), \tag{7}$$

$$\sigma_2 = Q(1,0) + Q(1,1). \tag{8}$$

Moreover, the current continuity condition gives

$$p_1 P(1,0) = p_2 Q(1,0). \tag{9}$$

Under the periodic boundary condition, the sum of the densities σ_1 and σ_2 should be twice the system density σ, i.e.,

$$\sigma_1 + \sigma_2 = 2\sigma. \tag{10}$$

Furthermore, by definition,

$$P(0,0) + P(0,1) + P(1,0) + P(1,1) = 1, \tag{11}$$

$$Q(0,0) + Q(0,1) + Q(1,0) + Q(1,1) = 1. \tag{12}$$

Now there are ten variables $P(0,0)$, $P(0,1)$, $P(1,0)$, $P(1,1)$, $Q(0,0)$, $Q(0,1)$, $Q(1,0)$, $Q(1,1)$, σ_1, σ_2 and ten equations (3)–(12). Thus, Eqs. (3)–(12) could be solved.

Nevertheless, following the mean-field analysis of Ref. [1], we can write out the master equation for other probabilities,

$$\frac{dP(1,0)}{dt} = -p_1 P(1,0) + p_2[P(0,1)P(0,0) + P(1,1)P(0,0) + P(1,1)P(0,0) + P(1,1)P(0,1)] = 0, \tag{13}$$

$$\frac{dP(0,1)}{dt} = p_1 P(1,0) - p_2[P(0,1)P(0,0) + P(0,1)P(0,1) + P(0,1)P(0,1) + P(1,1)P(0,1)] = 0, \tag{14}$$

$$\frac{dP(1,1)}{dt} = p_2[P(0,1)P(0,1) + P(1,1)P(0,1)] - p_2[P(1,1)P(0,0) + P(1,1)P(0,1)] = 0. \tag{15}$$

Equation (15) is identical to Eq. (3) and is not an independent equation. Moreover, Eq. (14) is also not an independent equation because it can be derived simply by substituting Eq. (3) into Eq. (13). However, it can be easily verified that Eq. (13) is a redundant equation because the solution to Eqs. (3)–(12) cannot meet the equation. Similarly, the counterpart of Eq. (13),

$$\frac{dQ(1,0)}{dt} = - p_2 Q(1,0) + p_1[Q(0,1)Q(0,0) + Q(1,1)Q(0,0) + Q(1,1)Q(0,0) + Q(1,1)Q(0,1)] = 0, \qquad (16)$$

is also a redundant equation.

We argue that this problem is due to the assumption that $P(x_i,x_{i+1},x_{i+2},x_{i+3})=P(x_i,x_{i+1})P(x_{i+2},x_{i+3})$ in Ref. [1]. Next we present a mean-field analysis in which there is no redundant equation. Note that an implicit difference between $P(x_i,x_{i+1},x_{i+2},x_{i+3})=P(x_i,x_{i+1})P(x_{i+2},x_{i+3})$ and Eq. (18) shown below is in how to express the probability of three consecutive sites. In the former case,

$$\begin{aligned} P(x_i,x_{i+1},x_{i+2}) &= P(0,x_i,x_{i+1},x_{i+2}) + P(1,x_i,x_{i+1},x_{i+2}) \\ &= [P(1,x_i) + P(0,x_i)]P(x_{i+1},x_{i+2}) \\ &= P(x_i)P(x_{i+1},x_{i+2}). \end{aligned}$$

In the latter case,

$$\begin{aligned} H(x_i,x_{i+1},x_{i+2}) &= H(1,x_i,x_{i+1},x_{i+2}) + H(0,x_i,x_{i+1},x_{i+2}) \\ &= H(1|\underline{x_i})H(x_i,x_{i+1})H(\underline{x_{i+1}}|x_{i+2}) \\ &\quad + H(0|\underline{x_i})H(x_i,x_{i+1})H(\underline{x_{i+1}}|x_{i+2}) \\ &= H(x_i,x_{i+1})H(\underline{x_{i+1}}|x_{i+2}) \\ &= H(x_i,x_{i+1})H(x_{i+1},x_{i+2})/[H(x_{i+1},0) \\ &\quad + H(x_{i+1},1)] = H(x_i,x_{i+1})H(x_{i+1},x_{i+2})/H(x_{i+1}) \\ &\neq H(x_i)H(x_{i+1},x_{i+2}). \end{aligned}$$

We denote $H(x_{i-1},x_i,x_{i+1},x_{i+2})$ as the probability of finding a p_2 site with occupancy x_{i-1} followed by a p_1 site with occupancy x_i, then followed by a p_2 site with occupancy x_{i+1}, then followed by a p_1 site with occupancy x_{i+2}. Based on this, the master equation for the probability $P(0,0)$ is

$$\frac{dP(0,0)}{dt} = - p_2[H(1,0,0,0) + H(1,0,0,1)] + p_2[H(1,0,1,0) + H(0,0,1,0)]. \qquad (17)$$

In the mean-field theory, $H(x_{i-1},x_i,x_{i+1},x_{i+2})$ is approximated by a product of overlapping two-site probabilities and conditional probabilities (see, e.g., Ref. [2]), i.e.,

$$H(x_{i-1},x_i,x_{i+1},x_{i+2}) = H(x_{i-1}|\underline{x_i})H(x_i,x_{i+1})H(\underline{x_{i+1}}|x_{i+2}). \qquad (18)$$

Here

$$H(x_{i-1}|\underline{x_i}) = \frac{H(x_{i-1},x_i)}{H(0,x_i) + H(1,x_i)},$$

$$H(\underline{x_{i+1}}|x_{i+2}) = \frac{H(x_{i+1},x_{i+2})}{H(x_{i+1},0) + H(x_{i+1},1)}.$$

From our definition of H, we know

$$H(x_{i-1},x_i) = Q(x_{i-1},x_i), H(x_i,x_{i+1}) = P(x_i,x_{i+1}), H(x_{i+1},x_{i+2}) = Q(x_{i+1},x_{i+2}). \qquad (19)$$

Therefore,

$$H(x_{i-1},x_i,x_{i+1},x_{i+2}) = Q(x_{i-1}|\underline{x_i})P(x_i,x_{i+1})Q(\underline{x_{i+1}}|x_{i+2}). \qquad (20)$$

Substituting Eq. (20) into Eq. (17), we have

$$\begin{aligned} \frac{dP(0,0)}{dt} &= - p_2[Q(1|\underline{0})P(0,0)Q(\underline{0}|0) + Q(1|\underline{0})P(0,0)Q(\underline{0}|1)] \\ &\quad + p_2[Q(1|\underline{0})P(0,1)Q(\underline{1}|0) + Q(0|\underline{0})P(0,1)Q(\underline{1}|0)] \\ &= - p_2[Q(1|\underline{0})P(0,0)] + p_2[P(0,1)Q(\underline{1}|0)] \\ &= - p_2\frac{Q(1,0)P(0,0)}{Q(1,0) + Q(0,0)} + p_2\frac{P(0,1)Q(1,0)}{Q(1,0) + Q(1,1)}. \end{aligned} \qquad (21)$$

Substituting Eqs. (7), (8), and (12) into Eq. (21), we have

$$\frac{dP(0,0)}{dt} = - p_2\frac{Q(1,0)P(0,0)}{1 - \sigma_1} + p_2\frac{P(0,1)Q(1,0)}{\sigma_2}. \qquad (22)$$

Therefore, in the steady state $\frac{dP(0,0)}{dt}=0$, we have

$$\frac{P(0,0)}{1 - \sigma_1} = \frac{P(0,1)}{\sigma_2}. \qquad (23)$$

Similarly, we have

$$\begin{aligned} \frac{dP(1,0)}{dt} &= - p_1[H(1,1,0,0) + H(1,1,0,1) \\ &\quad + H(0,1,0,0) + H(0,1,0,1)] \\ &\quad + p_2[H(1,0,0,0) + H(1,0,0,1) \\ &\quad + H(0,1,1,0) + H(1,1,1,0)] \\ &= - p_1P(1,0) + p_2\left[\frac{Q(1,0)P(0,0)}{1 - \sigma_1} + \frac{P(1,1)Q(1,0)}{\sigma_2}\right], \end{aligned} \qquad (24)$$

$$\begin{aligned} \frac{dP(0,1)}{dt} &= p_1[H(1,1,0,0) + H(1,1,0,1) \\ &\quad + H(0,1,0,0) + H(0,1,0,1)] \\ &\quad - p_2[H(1,0,1,0) + H(0,0,1,0) \\ &\quad + H(1,0,1,0) + H(1,0,1,1)] \\ &= p_1P(1,0) - p_2\left[\frac{P(0,1)Q(1,0)}{\sigma_2} + \frac{Q(1,0)P(0,1)}{1 - \sigma_1}\right], \end{aligned} \qquad (25)$$

$$\frac{dP(1,1)}{dt} = p_2[H(1,0,1,0) + H(1,0,1,1)] - p_2[H(1,1,1,0) + H(0,1,1,0)] = p_2[Q(1|\underline{0})P(0,1) - P(1,1)Q(\underline{1}|0)] = p_2\left[\frac{Q(1,0)P(0,1)}{1-\sigma_1} - \frac{P(1,1)Q(1,0)}{\sigma_2}\right]. \quad (26)$$

Substituting Eqs. (4), (5), (9), and (10) into Eqs. (24) and (25), it can be seen that the right-hand side of Eqs. (24) and (25) is zero. Therefore, Eqs. (24) and (25) are not independent equations. From Eqs. (4), (5), (11), and (23), $P(0,1)$ and $P(1,1)$ can be solved,

$$P(0,1) = \frac{(1-\sigma_1)\sigma_2}{1+\sigma_2-\sigma_1}, \quad (27)$$

$$P(1,1) = \frac{\sigma_2^2}{1+\sigma_2-\sigma_1}. \quad (28)$$

Substituting Eqs. (27) and (28) into Eq. (26), it is found that the right-hand side of Eq. (26) is zero. Therefore, Eq. (26) is also not an independent equation.

For $Q(x_i,x_j)$, one has the counterpart of Eq. (23),

$$\frac{Q(0,0)}{1-\sigma_2} = \frac{Q(0,1)}{\sigma_1}. \quad (29)$$

Therefore, in this mean-field method, we have ten variables $P(0,0)$, $P(0,1)$, $P(1,0)$, $P(1,1)$, $Q(0,0)$, $Q(0,1)$, $Q(1,0)$, $Q(1,1)$, σ_1, σ_2 and ten independent equations (4), (5), (7)–(12), (23), and (29). The equations can be solved, and surprisingly, the solution is the same as that presented in Ref. [1] without using Eqs. (13) and (16). This implies that the problem existing in the mean-field method presented in Ref. [1] might be resolved by discarding redundant equations. However, this introduces additional complexity into the method because one needs to judge which equations are redundant.

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